

Some Bounds for the Number of Blocks III

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Abstract

Let $\mathcal{D} = (\Omega, \mathcal{B})$ be a pair of v point set Ω and a set \mathcal{B} consists of k point subsets of Ω which are called blocks. Let d be the maximal cardinality of the intersections between the distinct two blocks in \mathcal{B} . The triple (v, k, d) is called the parameter of \mathcal{B} . Let b be the number of the blocks in \mathcal{B} . It is shown that inequality $\binom{v}{d+2i-1} \geq b \left\{ \binom{k}{d+2i-1} + \binom{k}{d+2i-2} \binom{v-k}{1} + \cdots + \binom{k}{d+i} \binom{v-k}{i-1} \right\}$ holds for each i satisfying $1 \leq i \leq k-d$, in the paper “Some Bounds for the Number of Blocks”, Europ. J. Combinatorics 22 (2001), 91–94, by R. Noda.

If b achieves the upper bound for some i , $1 \leq i \leq k-d$, then \mathcal{D} is called a $\beta(i)$ design. In the paper mentioned above, an upper bound and a lower bound, $\frac{(d+2i)(k-d)}{i} \leq v \leq \frac{(d+2(i-1))(k-d)}{i-1}$, for v of $\beta(i)$ design \mathcal{D} are given. In this paper we consider the cases when v does not achieve the upper bound or lower bound given above, and get new more strict bounds for v respectively. We apply this bound to the problem of the perfect e -codes in the Johnson scheme, and improve the bound given by Roos in the paper “A note on the existence of perfect constant weight codes”, Discrete Math. 47 (1983), 121–123.

Keywords: $\beta(i)$ design, $\beta(i)$ set, Johnson scheme, perfect e -code, diameter perfect code

2010 Mathematics Subject Classification: 05E30, 05B30

1 Introduction

This paper is a sequel to [9] and [10]. Let Ω be a v point set. We denote by $\binom{\Omega}{k}$ the family of all the k point subsets of Ω . A *design* is a pair $\mathcal{D} = (\Omega, \mathcal{B})$, where \mathcal{B} is a subset of $\binom{\Omega}{k}$. The elements in \mathcal{B} are called *blocks*. We denote the number of the blocks in \mathcal{B} by b . A design \mathcal{D} is called a t -(v, k, λ) design if for every t point subset of Ω there exist exactly λ blocks containing it. A t -($v, k, 1$) design is called a *Steiner system* and denoted by $S(t, k, v)$. A t -(v, k, λ) design or a Steiner system $S(t, k, v)$ is called *trivial* if $t = k$. In this paper we say that a design $\mathcal{D} = (\Omega, \mathcal{B})$, or the block set \mathcal{B} , has the *parameter* (v, k, d) if $\max\{|B \cap C| \mid B, C \in \mathcal{B}, B \neq C\} = d$. The number d plays an essential role in our paper. We assume $d < k < v$ throughout.

The following is a slightly revised version of a basic Proposition given in [9] for designs with the parameter (v, k, d) .

Proposition 0 ([9], see also [4])

Let $\mathcal{D} = (\Omega, \mathcal{B})$ be a design and (v, k, d) be the parameter of \mathcal{B} . Let $b = |\mathcal{B}|$. Then the following inequality

$$\binom{v}{d+2i-1} \geq b \left\{ \binom{k}{d+2i-1} + \binom{k}{d+2i-2} \binom{v-k}{1} + \cdots + \binom{k}{d+i} \binom{v-k}{i-1} \right\} \quad (1.1)$$

holds for any i satisfying $1 \leq i \leq \frac{v-d+1}{2}$. Moreover, equality in (1.1) holds for some i , $1 \leq i \leq \frac{v-d+1}{2}$, if and only if for any $(d+2i-1)$ point subset X of Ω , there exists a block $B \in \mathcal{B}$ with $|X \cap B| \geq d+i$.

Proposition 0 follows by counting in two ways the cardinality of the set $\{(X, B) \mid X \subset \Omega, |X| = d+2i-1, B \in \mathcal{B}, |X \cap B| \geq d+i\}$. Note that for any $(d+2i-1)$ point subset X , there exists at most one block B which satisfies $|X \cap B| \geq d+i$.

Definition 1 ($\beta(i)$ design)

We say that a design with the parameter (v, k, d) is a $\beta(i)$ design if the bound in (1.1) is achieved for some i , with $1 \leq i \leq \frac{v-d+1}{2}$.

Obviously if the bound in (1.1) is achieved for some i , with $1 \leq i \leq \frac{v-d+1}{2}$, then we must have

$$i \leq k - d. \quad (1.2)$$

A design $\mathcal{D} = (\Omega, \mathcal{B})$ with the parameter (v, k, d) is a $\beta(i)$ design if and only if for every $(d+2i-1)$ point subset $X \in \binom{\Omega}{d+2i-1}$ there exists a unique block $B \in \mathcal{B}$ with $|B \cap X| \geq d+i$. In other words, a design \mathcal{D} is a $\beta(i)$ design if and only if $\binom{\Omega}{d+2i-1}$ has the following partition parameterized by blocks in \mathcal{B} .

$$\binom{\Omega}{d+2i-1} = \bigcup_{B \in \mathcal{B}} \left\{ X \in \binom{\Omega}{d+2i-1} \mid |X \cap B| \geq d+i \right\}. \quad (1.3)$$

By Theorem 1 in [9] the *complementary design* of a $\beta(i)$ design with the parameter (v, k, d) is a $\beta(i')$ design with

$$i' = k - d - i + 1. \quad (1.4)$$

The parameter (v, k', d') of the complementary design of a $\beta(i)$ design with the parameter (v, k, d) is given by $k' = v - k$ and $d' = v - 2k + d$. Hence it follows that

$$k' - d' = k - d. \quad (1.5)$$

Steiner systems $S(t, k, v)$ are $\beta(1)$ designs (with $d = t - 1$) and conversely. By (1.4) the complementary designs of Steiner systems $S(d+1, k, v)$ are $\beta(k-d)$ designs and conversely. We call a $\beta(i)$ design *trivial* if $i = k - d$ holds. Only two non trivial $\beta(i)$ designs are known, the Steiner systems $S(5, 8, 24)$ ($i = 2$) and the complementary design of it ($i = 3$). These are also a $\beta(1)$ design and a $\beta(4)$ design respectively.

Remark

- (1) If \mathcal{D} is a $\beta(i)$ design with $i \geq 2$ which has the parameter $(v, k, d = 0)$, then \mathcal{D} is a Steiner system $S(1, k, 2k)$ with $k > 1$. In this case \mathcal{D} is a $\beta(i)$ design for every i with $i \leq k$.
- (2) If \mathcal{D} is a $\beta(i)$ design with the parameter $(v, k, d = k - 1)$, then $i = 1$ and \mathcal{D} is a trivial Steiner system $S(k, k, v)$.

Proof of the Remark above is straightforward and omitted.

In [9], Theorem 2 (1) the following inequalities for a $\beta(i)$ design \mathcal{D} are given.

$$\frac{(d + 2i)(k - d)}{i} \leq v \leq \frac{(d + 2(i - 1))(k - d)}{i - 1}. \quad (1.6)$$

Here v attains the upper bound in (1.6) if and only if \mathcal{D} is a $\beta(i - 1)$ design and v attains the lower bound in (1.6) if and only if \mathcal{D} is a $\beta(i + 1)$ design. Therefore if $d > 0$, then a $\beta(i)$ design can not be a $\beta(j)$ design at the same time for j with $|j - i| \geq 2$. The Steiner system $S(5, 8, 24)$ achieves the lower bound of (1.6) with $i = 1$ and the complementary design of it achieves the upper bound of (1.6) with $i = 4$. We remark that if a $\beta(i)$ design \mathcal{D} achieves the upper bound of (1.6) then the complementary design of \mathcal{D} achieves the lower bound of (1.6) and conversely.

In this paper we first give alternative bounds on the number of points in $\beta(i)$ designs if the upper or the lower bound given in (1.6) is not achieved (Theorems 1 and 2 in §2). In §3 by making use of Theorems 1 and 2 we give new bounds on v for a perfect e -code in the Johnson scheme $J(v, k)$ which improve the bound of Roos [12] (Theorem 3). In §4 we show that $k - d$ for a $\beta(i)$ design with parameter (v, k, d) is bounded in terms of linear expressions of i under some assumptions on $k - d$ and i (Theorems 4 and 5).

Our theorems are derived from the following basic propositions.

Proposition 6

Let \mathcal{D} be a $\beta(i)$ design with the parameter (v, k, d) , and let $c = k - d$. Assume that $i \geq 2$ and v does not achieve the upper bound in (1.6). Then we have $g(v - k, d, c, i) \geq 0$ where $g(x, d, c, i)$ is defined by

$$\begin{aligned} g(x, d, c, i) = & \\ & (i - 1)(i - 2)x^2 - (i - 1)\left(2(c - i + 1)d + 2c(i - 1) - 3i + 4\right)x \\ & + (c - i + 1)(c - i + 2)d^2 + (c - i + 1)\left((2i - 3)c + 3i - 4\right)d \\ & + (i - 1)(i - 2)c^2 + (i - 1)(3i - 4)c - 2(i - 1)^2(2i - 3). \end{aligned} \quad (1.7)$$

Moreover, $g(v - k, d, c, i) = 0$ if and only if $|\{B \in \mathcal{B} \mid |X \cap B| = d + i - 2\}|$ does not depend on the choice of $X \in S_1$, where S_1 is defined by

$$S_1 = \{X \subset \Omega \mid |X| = d + 2(i - 2), |X \cap B| \leq d + i - 2, \forall B \in \mathcal{B}\}. \quad (1.8)$$

Proposition 7

Let \mathcal{D} be a $\beta(i)$ design with the parameter (v, k, d) . Assume that $k - d \geq i + 1$ and v does not achieve the lower bound in (1.6). Then $h(v - k, d, c, i) \geq 0$ where $h(x, d, c, i)$ is defined by

$$\begin{aligned} h(x, d, c, i) = & i(i + 1)x^2 - i(2(c - i)d + 2ic + 3i + 1)x \\ & + (c - i)(c - i - 1)d^2 + (c - i)((2i + 1)c - 3i - 1)d \\ & + i(i + 1)c^2 - i(3i + 1)c + 2i^2(2i + 1). \end{aligned} \quad (1.9)$$

Moreover, $h(v - k, d, c, i) = 0$ if and only if $|\{B \in \mathcal{B} \mid |X \cap B| = d + i + 1\}|$ does not depend on the choice of $X \in S_2$, where S_2 is defined by

$$S_2 = \{X \subset \Omega \mid |X| = d + 2(i + 1), |X \cap B| \leq d + i + 1, \forall B \in \mathcal{B}\}. \quad (1.10)$$

It is shown in Lemma 2.1 that S_1 and S_2 are not empty under the assumptions of Propositions 6 and 7 respectively. Proofs of Propositions 6 and 7 are very long and involved, so we give them later in §5. In §6 we present some open problems in $\beta(i)$ designs.

Finally we would like to announce the following.

Theorem *Let \mathcal{D} be a $\beta(i)$ design with the parameter (v, k, d) . Then the following polynomial in t of degree $2(i - 1)$ has at least one positive integral zero and \mathcal{D} is a t -(v, k, λ) design for t which is the smallest of positive integral zeros of it.*

$$\sum_{j=0}^{i-1} (-1)^j \binom{t+1}{j} \sum_{s=0}^{i-j-1} \binom{k-t+j-1}{d-t+2i-2-s} \binom{v-k-j}{s}. \quad (1.11)$$

The proof of the above Theorem is given in a forthcoming paper [11].

Remark

The inequality given in Proposition 0 is also known by the researchers who studied maximal intersecting systems of finite sets which is originated by Erdős, Ko and Rado in 1961 [6]. Ahlswede-Aydinian-Khachatrian in 2001 [2] proved that the existence of the partition of $\binom{\Omega}{d+2i-1}$ given above is equivalent to an existence of the D -diameter perfect code with diameter $D = k - d + 1$. So the existence of a $\beta(i)$ design with the parameter (v, k, d) is equivalent to the existence of the D -diameter perfect code with diameter $D = k - d + 1$. They obtained upper and lower bound of the cardinality of a D -diameter perfect code which are equivalent to the bounds given in (1.6). We found out this fact very recently. For more information on intersecting systems of finite sets and perfect D -diameter codes please refer to [1], [2] and [5].

2 Some bounds on $v = |\Omega|$

In this section we give alternative bounds on v for $\beta(i)$ designs with the parameter (v, k, d) which do not achieve the bounds in (1.6). For this purpose we use the following two families of point subsets, S_1 and S_2 , defined in Proposition 6 (1.8) and Proposition 7 (1.10) respectively, for a $\beta(i)$ design $\mathcal{D} = (\Omega, \mathcal{B})$ with the parameter (v, k, d) .

$$\begin{aligned} S_1 &= \{X \subset \Omega \mid |X| = d + 2(i - 2), |X \cap B| \leq d + i - 2, \forall B \in \mathcal{B}\}. \\ S_2 &= \{X \subset \Omega \mid |X| = d + 2(i + 1), |X \cap B| \leq d + i + 1, \forall B \in \mathcal{B}\}. \end{aligned}$$

First we prove the following.

Lemma 2.1 *Let \mathcal{D} be a $\beta(i)$ design having the parameter (v, k, d) . Then*

- (1) S_1 is not empty if $i \geq 2$, and
- (2) S_2 is not empty if $k - d \geq i + 1$.

Proof

Let k' , d' and i' denote the parameters of the complementary design \mathcal{D}' of \mathcal{D} . Then

$$k' = v - k, \quad d' = v - 2k + d \quad \text{and} \quad i' = k - d - i + 1$$

by (1.4).

(1) If there exists a $d + 2(i - 2)$ point subset X such that $|X \cap B| = d + i - 2$ and $|X \cap (\Omega \setminus B)| = i - 2$ for some block B , then X belongs to S_1 . Therefore in order to prove (1) it suffices to show that $k \geq d + i - 2$ and $v - k \geq i - 2 \geq 0$. We have $k - d \geq i - 2$ by (1.2) and $i \geq 2$ by our assumption. Also since

$$(v - k) - (i - 2) = k' - (k' - d' - i' - 1) = d' + i' + 1 > 0,$$

we have $v - k > i - 2$.

(2) Similarly if there exists a $d + 2(i + 1)$ point subset Y such that $|Y \cap B| = d + i + 1$ and $|Y \cap (\Omega \setminus B)| = i + 1$ for some block B , then Y belongs to S_2 . We have $k \geq d + i + 1$ by our assumption. Therefore in order to prove (2) it suffices to show that $v - k \geq i + 1$. Since $d' \geq 0$ we have $v - k \geq k - d \geq i + 1$. ■

We can now state our results.

Theorem 1 *Let \mathcal{D} be a $\beta(i)$ design satisfying $i \geq 3$ and (v, k, d) be the parameter of \mathcal{B} with $v \geq 2k$. Let $c = k - d$. Assume that v does not achieve the upper bound in (1.6). Then we have*

$$v \leq k + \gamma_1 = c + d + \gamma_1, \tag{2.1}$$

where γ_1 is given by the following formula.

$$\begin{aligned} \gamma_1 = & \frac{(2c - 2i + 2)d + 2(i - 1)c - 3i + 4}{2(i - 2)} - \frac{1}{2(i - 1)(i - 2)} \times \\ & \left\{ 4(i - 1)(c - i + 1)(c - 2i + 3)d^2 + 4(i - 1)(3i - 4)(c - i + 1)(c - 2i + 3)d \right. \\ & \left. + (i - 1)^2 \left(4(2i - 3)c^2 - 4(3i - 4)(2i - 3)c + 16i^3 - 63i^2 + 80i - 32 \right) \right\}^{\frac{1}{2}}. \end{aligned} \quad (2.2)$$

Moreover, equality in (2.1) holds if and only if the number of the blocks B , for which $|B \cap X| = d + i - 2$, does not depend on the choice of $X \in S_1$.

Theorem 2

Let \mathcal{D} be a $\beta(i)$ design and (v, k, d) be the parameter of \mathcal{B} satisfying $v \leq 2k$ and $k - d \geq i + 1$. Let $c = k - d$. Assume that v does not achieve the lower bound in (1.6). Then we have

$$v \geq k + \gamma_2 = c + d + \gamma_2, \quad (2.3)$$

where γ_2 is given by the following formula.

$$\begin{aligned} \gamma_2 = & \frac{2(c - i)d + 2ic + 3i + 1}{2(i + 1)} + \frac{1}{2i(i + 1)} \times \\ & \left\{ -4i(c - i)(c - 2i - 1)d^2 - 4i(3i + 1)(c - i)(c - 2i - 1)d \right. \\ & \left. - i^2 \left(4(2i + 1)c^2 - 4(2i + 1)(3i + 1)c + 16i^3 + 15i^2 + 2i - 1 \right) \right\}^{\frac{1}{2}}. \end{aligned} \quad (2.4)$$

Moreover, equality in (2.3) holds if and only if the number of blocks B , for which $|B \cap X| = d + i + 1$, does not depend on the choice of $X \in S_2$.

In order to prove Theorems 1 and 2 we need the following.

Lemma 2.2 Let \mathcal{D} be a $\beta(i)$ design having the parameter (v, k, d) , and let $c = k - d$. Then we have the following.

- (1) If $v \geq 2k$, then $c \geq 2(i - 1)$ with equality if and only if \mathcal{D} is a $\beta(i - 1)$ design with $v = 2k$.
- (2) If $v \leq 2k$, then $c \leq 2i$ with equality if and only if \mathcal{D} is a $\beta(i + 1)$ design with $v = 2k$.
- (3) If $v = 2k$, then $2(i - 1) \leq c \leq 2i$ holds.

Proof

(1) Assume that $v \geq 2k$. Then by (1.6),

$$2k \leq v \leq \frac{(d + 2(i - 1))c}{i - 1} = \frac{dc}{i - 1} + 2c,$$

hence $2d \leq \frac{dc}{i-1}$. Therefore $2(i-1) \leq c$ with equality if and only if $v = 2k$ and the upper bound of (1.6) is attained. Recall that \mathcal{D} is a $\beta(i-1)$ design if and only if the upper bound of (1.6) is achieved.

(2) We can prove (2) in the same way as (1) by using the lower bound of (1.6).

(3) An immediate consequence of (1) and (2). ■

Let

$$x_1 = \frac{d(c-i)}{i} + c, \quad (2.5)$$

and let

$$x_2 = \frac{d(c-i+1)}{i-1} + c. \quad (2.6)$$

Then (1.6) implies

$$x_1 \leq v - k \leq x_2. \quad (2.7)$$

For the proof of Theorem 1, we use the upper bound on $v - k$, x_2 and for the proof of Theorem 2, we use the lower bound on $v - k$, x_1 as given above.

Proof of Theorem 1.

By assumption, \mathcal{D} is a $\beta(i)$ design and is not a $\beta(i-1)$ design with the parameter set (v, k, d) satisfying $i \geq 3$ and $v \geq 2k$. Then Proposition 6 implies that $g(v - k, d, c, i) \geq 0$. We show that $g(x_2, d, c, i) < 0$. We have

$$\begin{aligned} g(x_2, d, c, i) &= (i-1)(i-2) \left(\frac{d(c-i+1)}{i-1} + c \right)^2 \\ &\quad - (i-1) \left(2(c-i+1)d + 2c(i-1) - 3i + 4 \right) \left(\frac{d(c-i+1)}{i-1} + c \right) \\ &\quad + (c-i+1)(c-i+2)d^2 + (c-i+1) \left((2i-3)c + 3i - 4 \right) d \\ &\quad + (i-1)(i-2)c^2 + (i-1)(3i-4)c - 2(i-1)^2(2i-3) \\ &= -\frac{(d+i-1)(c-i+1)}{i-1} \left((c-2i+2)d + 2(i-1)c - 4i^2 + 10i - 6 \right). \end{aligned} \quad (2.8)$$

Since $v \geq 2k$ by our assumption, Lemma 2.2 (1) implies that $c \geq 2(i-1)$. Therefore

$$\begin{aligned} &(c-2i+2)d + 2(i-1)c - 4i^2 + 10i - 6 \\ &\geq 2(i-1)(2(i-1)) - 4i^2 + 10i - 6 = 2(i-1) > 0. \end{aligned} \quad (2.9)$$

By (2.8) and (2.9) we have

$$g(x_2, d, c, i) < 0. \quad (2.10)$$

Since $g(x, d, c, i)$ is a polynomial in x of degree 2 and the coefficient of x^2 is positive, $g(x, d, c, i)$ has two distinct real zeros. Let γ_1 be the smaller one. Since $g(v - k, d, c, i) \geq 0$

and $v - k \leq x_2$, we must have $v - k \leq \gamma_1$. Moreover the equality $v - k = \gamma_1$ holds if and only if $g(v - k, d, c, i) = 0$. Hence, Proposition 6 completes the proof of Theorem 1. ■

Proof of Theorem 2.

By assumption, \mathcal{D} is a $\beta(i)$ design and is not a $\beta(i + 1)$ design with the parameter set (v, k, d) satisfying $v \leq 2k$ and $c = k - d \geq i + 1$. Then Proposition 7 implies that $h(v - k, d, c, i) \geq 0$. We show that $h(x_1, d, c, i) < 0$. We have

$$\begin{aligned} h(x_1, d, c, i) &= i(i + 1) \left(\frac{d(c - i)}{i} + c \right)^2 - i \left(2(c - i)d + 2ic + 3i + 1 \right) \left(\frac{d(c - i)}{i} + c \right) \\ &\quad + (c - i)(c - i - 1)d^2 + (c - i) \left((2i + 1)c - 3i - 1 \right) d \\ &\quad + i(i + 1)c^2 - i(3i + 1)c + 2i^2(2i + 1) \\ &= \frac{(d + i)(c - i)}{i} \left((d + 2i)(c - 2i) - 2i \right). \end{aligned} \quad (2.11)$$

Since $v \leq 2k$ by our assumption, Lemma 2.2 (2) implies that $c \leq 2i$. Hence, it follows by (2.11) that

$$h(x_1, d, c, i) < 0. \quad (2.12)$$

Since $h(x, d, c, i)$ is a polynomial in x of degree 2 and the coefficient of x^2 is positive, $h(x, d, c, i)$ has two distinct real zeros. Let γ_2 be the larger one. Since $h(v - k, d, c, i) \geq 0$ and $v - k \geq x_1$, we must have $v - k \geq \gamma_2$. Moreover the equality, $v - k = \gamma_2$, holds if and only if $h(v - k, d, c, i) = 0$. Hence, Proposition 7 completes the proof of Theorem 2. ■

Remark

- (1) If \mathcal{D} achieves the upper bound in (1.6), then for every $X \in S_1$ there exist $\frac{v-d-2(i-2)}{k-d-i+2}$ blocks in \mathcal{B} which have $d + i - 2$ points in common with X .
- (2) If \mathcal{D} achieves the lower bound in (1.6), then for every $X \in S_2$ there exist $\frac{d+2(i+1)}{i+1}$ blocks in \mathcal{B} which have $d + i + 1$ points in common with X .

3 Application to perfect e -codes

The *Johnson graph* (or the *Johnson scheme*) $J(v, k)$ is a graph whose set of vertices is $\binom{\Omega}{k}$ for a v point set Ω . Two vertices x and y are *adjacent* if and only if $|x \cap y| = k - 1$. The *distance* $d(x, y)$ between two vertices x and y is the *length* of the *shortest path* which connects these vertices, that is, $d(x, y) = k - |x \cap y|$. A *code* \mathcal{C} in $J(v, k)$ is a subset of $\binom{\Omega}{k}$. The *minimum distance* of \mathcal{C} is defined by $d(\mathcal{C}) = \min\{d(x, y) | x \neq y, x, y \in \mathcal{C}\}$. A code \mathcal{C} is called a *perfect e -code* in $J(v, k)$ if the e -spheres with centers at the code words of \mathcal{C} form a partition of $\binom{\Omega}{k}$. In other words, \mathcal{C} is a perfect e -code if for each element $x \in \binom{\Omega}{k}$ there exists a unique element $c \in \mathcal{C}$ such that $d(x, c) \leq e$. Clearly, the minimum distance of a perfect e -code is $2e + 1$.

Then it is easily seen that a $\beta(i)$ set \mathcal{B} having the parameter (v, k, d) with $k - d = 2i - 1$ is a perfect $(i - 1)$ -code in the Johnson scheme $J(v, k)$ and conversely. As for a perfect e -code in $J(v, k)$, Roos [12] proved the inequality $v \leq (k - 1)(2e + 1)/e$. This corresponds to the upper bound of (1.6). T. Etzion and M. Schwartz ([7], Theorem 13) showed that the bound of Roos is not achievable. For the details of perfect e -codes in the Johnson scheme $J(v, k)$ please refer to [8].

By making use of the same argument as employed in the proofs of Theorems 1 and 2 we prove the following theorem which improves the upper bound of Roos.

Theorem 3

Let e be an integer satisfying $e \geq 2$ and \mathcal{C} be a perfect e -code in the Johnson scheme $J(v, k)$. Then we have the following inequality.

$$\frac{2(e+1)k}{e+2} + \frac{7e+6}{2(e+2)} + \frac{\sqrt{A_2}}{2(e+1)(e+2)} \leq v \leq \frac{2ke}{e-1} - \frac{7e+1}{2(e-1)} - \frac{\sqrt{A_1}}{2e(e-1)}, \quad (3.1)$$

where

$$A_1 = e \left\{ 8(e+1) \left(k - \frac{e+3}{2} \right)^2 - (e+2)(e-1)^2 \right\}$$

and

$$A_2 = (e+1) \left\{ 8e \left(k - \frac{e-2}{2} \right)^2 - (e-1)(e+2)^2 \right\}.$$

Moreover, v achieves the upper bound in (3.1) if and only if for any word y of length v and of weight $k - 3$, for which every $u \in \mathcal{C}$ satisfies $|y \cap u| \leq k - e - 2$, the number of $u \in \mathcal{C}$ which satisfies $|u \cap y| = k - e - 2$, is invariant, that is, independent of the choice of such y . Also, the lower bound holds if and only if for every word y of length v and of weight $k + 3$, for which every $u \in \mathcal{C}$ satisfies $|y \cap u| \leq k - e + 1$, the number of $u \in \mathcal{C}$, which satisfies $|u \cap y| = k - e + 1$, is invariant.

Proof

Perfect e -codes in the Johnson scheme $J(v, k)$ can be regarded as block set of $\beta(i)$ designs having the parameter (v, k, d) , with $i = e + 1$ and $d = k - 2e - 1 = k - 2i + 1$. Then since $2(i - 1) < c = k - d < 2i$, we have $g(x_2, d, c, i) < 0$ by (2.10) and $h(x_1, d, c, i) < 0$ by (2.12). Hence the same argument as given in the proofs of Theorems 1 and 2 yields Theorem 3. ■

4 Bounds on $k - d$

In this section we give bounds on $k - d$ of a $\beta(i)$ design \mathcal{D} in terms of i . These bounds are very important and useful for the classification of $\beta(i)$ designs.

Theorem 4

Let \mathcal{D} be a $\beta(i)$ design with $i \geq 3$ and let (v, k, d) be the parameter of \mathcal{D} , where $v \geq 2k$. Then the following hold.

(1) If \mathcal{D} is not a $\beta(i-1)$ design, then

$$2(i-1) \leq k-d < \frac{i(3i-2+\sqrt{i^2+12i-12})}{2(i-2)}. \quad (4.1)$$

In particular

$$2(i-1) \leq k-d \leq 2i+6 \quad (4.2)$$

for $i \geq 8$.

(2) If \mathcal{D} is a $\beta(i-1)$ design with $d > 0$, then

$$2(i-1) \leq k-d \leq 2i+4 \quad (4.3)$$

for $i \geq 9$.

Proof

(1) Let $g(x, d, c, i)$ be the polynomial defined in Proposition 6. Let $x_1 = \frac{d(c-i)}{i} + c$ and $x_2 = \frac{d(c-i+1)}{i-1} + c$ be the lower bound and the upper bound on $v-k$ defined in (2.5) and (2.6) respectively. We have $g(x_2, d, c, i) < 0$ by (2.10) and $g(v-k, d, c, i) \geq 0$ by Proposition 6. Then since the coefficient of x^2 in the quadratic polynomial $g(x, v, k, d)$ is positive and $x_1 \leq v-k$, it follows that $g(x_1, d, c, i) \geq 0$. By substituting x_1 in $g(x, d, c, i)$ we have

$$\begin{aligned} \frac{1}{i^2} g(x_1, d, c, i) = & -\left((i-2)c^2 - (3i^2 - 2i)c + 2i^2(i-1)\right)d^2 \\ & -i\left((3i-2)c - 3i^2 + 4i\right)\left(c - 2(i-1)\right)d - 2i^2(i-1)(c-i+1)(c-2i+3), \\ & c - 2i + 3 \geq 0. \end{aligned} \quad (4.4)$$

Since $i > 2$ and $c \geq 2(i-1)$, it follows that

$$\begin{aligned} (3i-2)c - 3i^2 + 4i & \geq (3i-2)(2i-2) - 3i^2 + 4i = 3i(i-2) + 4 > 0, \\ c - 2(i-1) & \geq 0, \\ 2i^2(i-1)(c-i+1)(c-2i+3) & \geq 2i^2(i-1)^2 > 0. \end{aligned} \quad (4.5)$$

By (4.4) and (4.5), we have

$$(i-2)c^2 - (3i^2 - 2i)c + 2i^2(i-1) < 0. \quad (4.6)$$

This implies

$$2(i-1) \leq c < \frac{i(3i-2+\sqrt{i^2+12i-12})}{2(i-2)} \quad (4.7)$$

It is easy to see that if $i \geq 8$, then $\frac{i(3i-2+\sqrt{i^2+12i-12})}{2(i-2)} < 2i+7$ holds. This completes the proof of (1).

(2) As mentioned in the Introduction, if \mathcal{D} is a $\beta(i)$ design and a $\beta(i-1)$ design with $d > 0$, then \mathcal{D} cannot be a $\beta(i-2)$ design. Therefore we can apply (1) to (2) and we have $c \leq 2(i-1) + 6$ for $i-1 \geq 8$. This completes the proof of Theorem 4. \blacksquare

Theorem 5

Let \mathcal{D} be a $\beta(i)$ design with the parameter (v, k, d) . Assume that $v \leq 2k$ and $c = k - d \geq i + 2$. Then the following hold.

(1) If \mathcal{D} is not a $\beta(i+1)$ design and if $i \geq 18$, then

$$\frac{(i-1)(3i-1+\sqrt{i^2-14i+1})}{2(i+1)} < k-d \leq 2i. \quad (4.8)$$

In particular

$$2i-8 \leq k-d \leq 2i.$$

(2) If \mathcal{D} is not a $\beta(i+1)$ design and if $c = k - d \geq 3$, then

$$\frac{(i-1)(3i-1+\sqrt{i^2-14i+1})}{2(i+1)} < k-d \leq 2i, \quad (4.9)$$

for $i \geq 15$. In particular

$$2i-8 \leq k-d \leq 2i,$$

if $i \geq 16$.

(3) If \mathcal{D} is a $\beta(i+1)$ design with $d > 0$ and if $i \geq 17$, then

$$2i-6 \leq k-d \leq 2i. \quad (4.10)$$

Proof

Our assumption $v \leq 2k$ and Lemma 2.2 imply that $c \leq 2i$. Let \mathcal{D}' be the complementary design of \mathcal{D} . Then \mathcal{D}' is a $\beta(i')$ design with the parameter (v, k', d') where

$$i' = k - d - i + 1 = c - i + 1, \quad k' = v - k, \quad d' = v - 2k + d$$

(see (1.4)). Then $v \leq 2k$ and $k - d \geq i + 2$ imply

$$v \geq 2k' \quad \text{and} \quad i' = k - d - i + 1 \geq 3.$$

Therefore we can apply Theorem 2 to \mathcal{D}' .

(1) If \mathcal{D} is not a $\beta(i+1)$ design, then \mathcal{D}' is not a $\beta(k-d-(i+1)+1)$ design, i.e. \mathcal{D}' is a $\beta(i')$, but not a $\beta(i'-1)$ design. Hence we can use the inequality (4.6) for \mathcal{D}' . By substituting $i' = c - i + 1$ and $c' = c$, we obtain

$$(i'-2)c^2 - 3(i'^2 - 2i')c + 2i'^2(i'-1) < 0$$

and so

$$(c - i - 1)c^2 - 3((c - i + 1)^2 - 2(c - i + 1))c + 2(c - i + 1)^2(c - i) < 0,$$

hence

$$(i + 1)c^2 - (3i - 1)(i - 1)c + 2i(i - 1)^2 > 0. \quad (4.11)$$

By (4.11), for $i \geq 14$ we have

$$c > \frac{(i - 1)(3i - 1 + \sqrt{i^2 - 14i + 1})}{2(i + 1)} \quad \text{or} \quad c < \frac{(i - 1)(3i - 1 - \sqrt{i^2 - 14i + 1})}{2(i + 1)}.$$

Assume that the latter holds. Then by our assumption on c we have

$$i + 2 \leq c < \frac{(i - 1)(3i - 1 - \sqrt{i^2 - 14i + 1})}{2(i + 1)},$$

and so

$$(i - 1)(\sqrt{i^2 - 14i + 1}) < i^2 - 10i - 3,$$

hence

$$(i + 1)(i^2 - 17i - 2) < 0.$$

This implies $i \leq 17$. Therefore if $i \geq 18$, we have $c > \frac{(i-1)(3i-1+\sqrt{i^2-14i+1})}{2(i+1)}$, hence $c \geq 2i - 8$.

(2) As in the above the inequality $i + 3 \leq \frac{(i-1)(3i-1-\sqrt{i^2-14i+1})}{2(i+1)}$ gives $(i + 1)(i^2 - 14i - 3) < 0$.

This implies $i \leq 14$. Therefore if $i \geq 15$ we have $c > \frac{(i-1)(3i-1+\sqrt{i^2-14i+1})}{2(i+1)}$ and hence we have $c \geq 2i - 8$ if $i \geq 16$.

(3) If \mathcal{D} is a $\beta(i + 1)$ design then \mathcal{D} is not a $\beta(i + 2)$ design. Hence by (1) we have $2(i + 1) - 8 \leq c \leq 2i$ for $i + 1 \geq 18$. This completes the proof of Theorem 5. \blacksquare

5 Proof of Basic Propositions

In this section we give the proofs of the basic Propositions 6 and 7 which are stated in §1 and already used in the proofs of our main results in the previous sections.

The proof needs lots of preparations. In the statements of Propositions 6 and 7, we defined two families of subsets of Ω , S_1 and S_2 . We state the definitions of S_1 and S_2 here again.

$$\begin{aligned} S_1 &= \{X \subset \Omega \mid |X| = d + 2(i - 2), |X \cap B| \leq d + i - 2, \forall B \in \mathcal{B}\}. \\ S_2 &= \{X \subset \Omega \mid |X| = d + 2(i + 1), |X \cap B| \leq d + i + 1, \forall B \in \mathcal{B}\}. \end{aligned}$$

By Lemma 2.1, S_1 and S_2 are not empty under the assumption of Propositions 6 and 7.

We first introduce the following combinatorial formula which we use later in the proof of Proposition 5.5.

Lemma 5.1 *Let m , s and j be integers satisfying $m > 0$ and $s \geq j > 0$. Then the following holds.*

$$\binom{m}{s} + \binom{m}{s-1} \binom{j}{1} + \binom{m}{s-2} \binom{j}{2} + \cdots + \binom{m}{s-j} \binom{j}{j} = \frac{\binom{m+j}{j} \binom{m}{s-j}}{\binom{s}{j}}. \quad (5.1)$$

We will give the proof of Lemma 5.1 later, at the end of this section.

Now we are ready to start the proofs of Propositions 6 and 7. We first prove the following.

Proposition 5.2 *Let $\mathcal{D} = (\Omega, \mathcal{B})$ be a $\beta(i)$ design with the parameter (v, k, d) and $B \in \mathcal{B}$ arbitrarily fixed. Let $\mu_d = |\{C \in \mathcal{B} \mid |C \cap B| = d\}|$ and $c = k - d$. Then*

$$\mu_d = \frac{\binom{v-k}{i} \binom{k}{d+i-1}}{\binom{c}{i} \binom{c}{i-1}} \quad (5.2)$$

holds. In particular μ_d is independent of the choice of $B \in \mathcal{B}$.

Proof

We count the cardinality of the following set in two different ways.

$$\left\{ (X, C) \mid \begin{array}{l} X \subset \Omega, C \in \mathcal{B}, |X \cap B| = d + i - 1, |X \cap (\Omega \setminus B)| = i, \\ |X \cap B \cap C| = d, |X \cap (\Omega \setminus B) \cap C| = i \end{array} \right\}. \quad (5.3)$$

Let $X \subset \Omega$ be any fixed subset of Ω satisfying $|X \cap B| = d + i - 1$ and $|X \cap (\Omega \setminus B)| = i$. Since \mathcal{D} is a $\beta(i)$ design, there exists a unique block $C \in \mathcal{B}$ satisfying $|X \cap C| \geq d + i$ (see Definition 1 and (1.3)). Then since $|B \cap C| \leq d$, we must have $|X \cap B \cap C| = d$ and $|X \cap (\Omega \setminus B) \cap C| = i$. Hence the cardinality of the set defined in (5.3) equals $\binom{k}{d+i-1} \binom{v-k}{i}$. On the other hand, for any block C satisfying $|B \cap C| = d$, the number of subsets $X \subset \Omega$ satisfying $|X \cap B| = d + i - 1$, $|X \cap B \cap C| = d$ and $|X \cap (\Omega \setminus B) \cap C| = i$ equals $\binom{k-d}{i-1} \binom{k-d}{i}$. Hence the cardinality of the set defined in (5.3) equals $\mu_d \binom{k-d}{i-1} \binom{k-d}{i}$. This completes the proof of Proposition 5.2. ■

Proposition 5.3 *Let $\mathcal{D} = (\Omega, \mathcal{B})$ be a $\beta(i)$ design with the parameter (v, k, d) . Let $i \geq 2$, $c = k - d$, $b = |\mathcal{B}|$ and $n = |S_1|$, where S_1 is as defined in (1.8). Then we have*

$$\frac{b}{n} \binom{c+d}{d+i-2} \binom{v-k}{i-2} \leq 1 + \frac{(i-1)(v-k-i+1)(v-k-i+2)}{(d+i-1)(c-i+1)(c-i+2)}. \quad (5.4)$$

Moreover, equality in (5.4) holds if and only if $|\{B \mid |B \cap X| = d + i - 2\}|$ does not depend on the choice of $X \in S_1$.

Proof

For $X \in S_1$ let α_X be defined as follows.

$$\alpha_X = |\{B \in \mathcal{B} \mid |X \cap B| = d + i - 2\}|.$$

Let $P = \sum_{X \in S_1} \alpha_X$ and $Q = \sum_{X \in S_1} \alpha_X(\alpha_X - 1)$. Then

$$0 \leq \sum_{X \in S_1} \left(\alpha_X - \frac{P}{n} \right)^2 = \sum_{X \in S_1} \left(\alpha_X^2 - \frac{2P}{n} \alpha_X + \frac{P^2}{n^2} \right) = \sum_{X \in S_1} \alpha_X^2 - \frac{P^2}{n} = Q + P - \frac{P^2}{n}$$

implies

$$\frac{P}{n} \leq 1 + \frac{Q}{P}, \quad (5.5)$$

where equality holds if and only if $\alpha_X = \frac{P}{n}$ (constant), i.e. $|\{B \mid |B \cap X| = d + i - 2\}|$ does not depend on the choice of $X \in S_1$. We remark that if a $\beta(i)$ design \mathcal{D} is also a $\beta(i-1)$ design, then α_X is a constant given by $\alpha_X = \frac{v-d-2(i-2)}{k-d-i+2} = \frac{v-k+c-2(i-2)}{c-i+2}$.

Next we express P and Q in terms of the parameters v , b , k , d , and i . As for P , by counting the cardinality of the following set,

$$\{(X, B) \mid B \in \mathcal{B}, X \in S_1, |X \cap B| = d + i - 2\},$$

in two ways, we obtain the following equality.

$$P = \sum_{X \in S_1} \alpha_X = b \binom{k}{d+i-2} \binom{v-k}{i-2}. \quad (5.6)$$

Note that if a $d + 2(i-2)$ design X has $d + i - 2$ points in common with some block then $X \in S_1$. It is easy to see that if two distinct blocks B and C satisfy $|X \cap B| = |X \cap C| = d + i - 2$ then $|X \cap B \cap C| = d$ holds. Therefore by counting the cardinality of the following set in two ways

$$\{(X, B, C) \mid B, C \in \mathcal{B}, B \neq C, X \in S_1, |X \cap B| = |X \cap C| = d + i - 2\},$$

we obtain the following equation.

$$Q = \sum_{X \in S_1} \alpha_X(\alpha_X - 1) = b\mu_d \binom{k-d}{i-2}^2 = b\mu_d \binom{c}{i-2}^2.$$

where μ_d is defined in Proposition 5.2. Then (5.2) implies

$$Q = b \frac{\binom{v-k}{i} \binom{k}{d+i-1}}{\binom{c}{i} \binom{c}{i-1}} \binom{c}{i-2}^2. \quad (5.7)$$

Then (5.5), (5.6) and (5.7) imply Proposition 5.3. ■

Proposition 5.4 Let $b = |B|$, $n = |S_1|$ and $c = k - d$. Then we have the following formula.

$$n = \frac{bF(v-k)}{\prod_{l=2}^4 (v-k+c-2i+l)}, \quad (5.8)$$

where $F(x)$ is defined by

$$F(x) = \binom{x}{i-2} (px^2 + qx + r),$$

$$p = -(i-2) \binom{k}{d+i-1}, \quad (5.9)$$

$$q = -\frac{1}{i-1} \binom{k}{d+i-1} \left\{ d^3 - (k-6i+7)d^2 - \left((5i-6)k - (i-1)(9i-11) \right) d - (i-1)(4i-5)k + (i-1)(2i^2 - 2i - 1) \right\}, \quad (5.10)$$

and

$$r = \frac{1}{i-1} \binom{k}{d+i-1} \left\{ (2i-3)d^3 - (3i-4)(k-3i+4)d^2 + \left((i-1)k^2 - (9i^2 - 23i + 15)k + (i-1)(12i^2 - 33i + 23) \right) d + (i-1)^2 k^2 - (i-1)(6i^2 - 16i + 11)k + 2(2i-3)(i-1)^3 \right\}. \quad (5.11)$$

Proof

For a $d+2(i-2)$ set X , there exists at most one block B satisfying $|B \cap X| \geq d+i-1$. Therefore, we have

$$n = \binom{v}{d+2i-4} - b \sum_{j=0}^{i-3} \binom{k}{d+2i-4-j} \binom{v-k}{j}. \quad (5.12)$$

Since \mathcal{D} is a $\beta(i)$ design, the equality in (1.1) implies the following.

$$b = \frac{\binom{v}{d+2i-1}}{\binom{k}{d+2i-1} + \binom{k}{d+2i-2} \binom{v-k}{1} + \cdots + \binom{k}{d+i} \binom{v-k}{i-1}}. \quad (5.13)$$

Therefore, we have

$$\begin{aligned} \binom{v}{d+2(i-2)} &= \frac{(d+2i-1)(d+2i-2)(d+2i-3)}{(v-(d+2i-4))(v-(d+2i-3))(v-(d+2i-2))} \binom{v}{d+2i-1} \\ &= \frac{(d+2i-1)(d+2i-2)(d+2i-3)}{(v-(d+2i-4))(v-(d+2i-3))(v-(d+2i-2))} \times \\ &= b \left(\binom{k}{d+2i-1} + \binom{k}{d+2i-2} \binom{v-k}{1} + \cdots + \binom{k}{d+i} \binom{v-k}{i-1} \right). \end{aligned} \quad (5.14)$$

Then (5.12) and (5.14) imply the following equality,

$$\begin{aligned}
n = & \frac{(d+2i-1)(d+2i-2)(d+2i-3)}{(v-k+c-2i+4)(v-k+c-2i+3)(v-k+c-2i+2)} \times \\
& b \left(\binom{k}{d+2i-1} + \binom{k}{d+2i-2} \binom{v-k}{1} + \cdots + \binom{k}{d+i} \binom{v-k}{i-1} \right) \\
& - b \left(\binom{k}{d+2i-4} + \binom{k}{d+2i-3} \binom{v-k}{1} + \cdots + \binom{k}{d+i-1} \binom{v-k}{i-3} \right). \tag{5.15}
\end{aligned}$$

Therefore, if we define a polynomial $F(x)$ in x by

$$\begin{aligned}
F(x) = & \left\{ (d+2i-1)(d+2i-2)(d+2i-3) \times \right. \\
& \left(\binom{k}{d+2i-1} + \binom{k}{d+2i-2} \binom{x}{1} + \cdots + \binom{k}{d+i} \binom{x}{i-1} \right) \\
& - (x+c-2i+4)(x+c-2i+3)(x+c-2i+2) \times \\
& \left. \left(\binom{k}{d+2i-4} + \binom{k}{d+2i-5} \binom{x}{1} + \cdots + \binom{k}{d+i-1} \binom{x}{i-3} \right) \right\}, \tag{5.16}
\end{aligned}$$

we obtain

$$n = \frac{bF(v-k)}{(v-k+c-2i+4)(v-k+c-2i+3)(v-k+c-2i+2)}. \tag{5.17}$$

In the following we will prove that $F(x)$ has the expression given in the statement of Proposition 5.4. For this purpose we define a polynomial $G(x)$ in x of degree $i+2$ by

$$\begin{aligned}
G(x) = & (d+2i-1)(d+2i-2)(d+2i-3) \times \\
& \left\{ \binom{k}{d+2i-1} + \binom{k}{d+2i-2} \binom{x}{1} + \cdots + \binom{k}{d+i} \binom{x}{i-1} \right\} \\
& - (x+c-2i+4)(x+c-2i+3)(x+c-2i+2) \times \\
& \left\{ \binom{k}{d+2i-4} + \binom{k}{d+2i-5} \binom{x}{1} + \cdots + \binom{k}{d+i-3} \binom{x}{i-1} \right\}. \tag{5.18}
\end{aligned}$$

Then we have

$$\begin{aligned}
F(x) = & G(x) + (x+c-2i+4)(x+c-2i+3)(x+c-2i+2) \times \\
& \left\{ \binom{c+d}{d+i-2} \binom{x}{i-2} + \binom{c+d}{d+i-3} \binom{x}{i-1} \right\}. \tag{5.19}
\end{aligned}$$

We claim that the following proposition holds.

Proposition 5.5 (1) $G(j) = 0$ for any integer j with $0 \leq j \leq i-1$.

(2) $F(j) = 0$ for any integer j with $0 \leq j \leq i-3$.

$$(3) \quad F(i-2) = \frac{(c+d)!}{(d+i-2)!(c-i-1)!},$$

$$F(i-1) = (c-i+3)(c-i+2)(c-i+1) \left\{ \binom{c+d}{d+i-2}(i-1) + \binom{c+d}{d+i-3} \right\}.$$

Proof.

(1) Let j be an integer satisfying $0 \leq j \leq i-1$. Then by (5.1) we have

$$\begin{aligned} G(j) &= (d+2i-1)(d+2i-2)(d+2i-3) \frac{\binom{c+d+j}{j} \binom{c+d}{d+2i-1-j}}{\binom{d+2i-1}{j}} \\ &\quad - (j+c-2i+4)(j+c-2i+3)(j+c-2i+2) \frac{\binom{c+d+j}{j} \binom{c+d}{d+2i-4-j}}{\binom{d+2i-4}{j}} \\ &= \binom{c+d+j}{j} \left(\frac{(c+d)!j!}{(d+2i-4)!(c-2i+1+j)!} - \frac{(c+d)!j!}{(c-2i+1+j)!(d+2i-4)!} \right) \\ &= 0. \end{aligned} \tag{5.20}$$

(2) For $0 \leq j \leq i-3$, (1) and (5.19) imply $F(j) = G(j) = 0$.

(3) Since $G(i-2) = G(i-1) = 0$, (5.19) implies

$$\begin{aligned} F(i-2) &= (c-i+2)(c-i+1)(c-i) \binom{c+d}{d+i-2} = \frac{(c+d)!}{(c-i-1)!(d+i-2)!}. \\ F(i-1) &= (c-i+3)(c-i+2)(c-i+1) \left\{ \binom{c+d}{d+i-2}(i-1) + \binom{c+d}{d+i-3} \right\} \\ &= \frac{((i-1)c - i^2 + d + 5(i-1))(c+d)!}{(c-i)!(d+i-2)!}. \end{aligned}$$

■

Now we are ready to finish the proof of Proposition 5.4. By (5.16), $F(x)$ is a polynomial in x of degree i and part (2) of Proposition 5.5 shows that $0, 1, \dots, i-3$ are zeros of $F(x)$. Hence $F(x)$ is expressed by

$$F(x) = \binom{x}{i-2} (px^2 + qx + r), \tag{5.21}$$

where p, q , and r are some rational expressions of c , d , and i . Moreover, since the coefficient of x^i in $F(x)$ is $-\binom{c+d}{d+i-1} \frac{1}{(i-3)!}$, we must have

$$p = -\binom{c+d}{d+i-1} (i-2). \tag{5.22}$$

Then, part (3) of Proposition 5.5 and (5.21) imply

$$(c-i+2)(c-i+1)(c-i) \binom{c+d}{d+i-2} = p(i-2)^2 + q(i-2) + r, \tag{5.23}$$

and

$$\begin{aligned} &(c-i+3)(c-i+2)(c-i+1) \left\{ \binom{c+d}{d+i-2}(i-1) + \binom{c+d}{d+i-3} \right\} \\ &= (i-1)(p(i-1)^2 + q(i-1) + r). \end{aligned} \tag{5.24}$$

Then (5.22), (5.23) and (5.24) imply the formula of $F(x)$ given in Proposition 5.4. \blacksquare

Proof of Proposition 6

Proposition 5.3 and (5.8) imply the following inequality

$$\begin{aligned} & \frac{(v-k+c-2i+4)(v-k+c-2i+3)(v-k+c-2i+2)}{F(v-k)} \binom{c+d}{d+i-2} \binom{v-k}{i-2} \\ & \leq 1 + \frac{(i-1)(v-k-i+1)(v-k-i+2)}{(d+i-1)(c-i+1)(c-i+2)}. \end{aligned} \quad (5.25)$$

Then the formula of $F(x)$ given in Proposition 5.4 implies

$$\begin{aligned} 0 & \leq (p(v-k)^2 + q(v-k) + r) \left(1 + \frac{(i-1)(v-k-i+1)(v-k-i+2)}{(d+i-1)(c-i+1)(c-i+2)} \right) \\ & - (v-k+c-2i+4)(v-k+c-2i+3)(v-k+c-2i+2) \binom{c+d}{d+i-2}. \\ & = \frac{(c+d)!(v-k-i+2)(cd-c+(c-d)i+d-(i-1)(v-k))}{(c-i+2)!(d+i-1)!(d+i-1)(c-i+1)(i-1)} g(v-k, d, c, i), \\ & = \frac{(c+d)!(v-k-i+2)((d+2i-2)c-(i-1)v)}{(c-i+2)!(d+i-1)!(d+i-1)(c-i+1)(i-1)} g(v-k, d, c, i), \end{aligned} \quad (5.26)$$

where $g(x, d, c, i)$ is defined by (1.7). Since v does not achieve the upper bound of (1.6), by the assumption, we have $(d+2i-2)c-(i-1)v > 0$. Hence by (5.26) we obtain $g(v-k, d, c, i) \geq 0$.

Finally, by Proposition 5.3 the equality $g(v-k, d, c, i) = 0$ holds if and only if $|\{B \mid |B \cap X| = d+i-2\}|$ does not depend on the choice of $X \in S_1$. This completes the proof of Proposition 6. \blacksquare

Remark

- (1) If $i = 2$, then the inequality $g(x, d, c, i) \geq 0$ is reduced to the inequality $x \leq (cd+2)/2$. The complementary designs of Steiner systems $S(t, t+1, 2t+3)$ achieve this bound. In Theorem 2 in [9], the parameters of non trivial $\beta(2)$ sets are expressed by means of two parameters and we see that there exists no non trivial $\beta(2)$ design in which $g(v-k, d, c, 2) = 0$ holds.
- (2) The parameter $(v = 2k, k = d + 2i - 3, d)$ satisfies $g(v-k, d, c, i) = 0$. However by Lemma 2.2 there exists no $\beta(i)$ design having this parameter.

Proof of Proposition 7

By assumption, v does not achieve the lower bound in (1.6), i.e. $v > (d+2i)(k-d)/i$. By Theorem 1 [9], the complementary design \mathcal{D}' of \mathcal{D} is a $\beta(i')$ design with the parameter (v, k', d') , where

$$i' = k - d - i + 1 \geq 2, \quad k' = v - k, \quad d' = v - 2k + d.$$

Since \mathcal{D} is not a $\beta(i+1)$ design, \mathcal{D}' is not a $\beta(k-d-(i+1)+1) = \beta(i'-1)$ design. Then Proposition 6 implies $g(v-k', d', c', i') = g(c+d, v-k-c, c, c-i+1) \geq 0$. On the other hand we have

$$\begin{aligned}
& g(c+d, x-c, c, c-i+1) \\
&= (c-i)(c-i-1)(c+d)^2 - (c-i)(c+d) \left(2(x-c)(2c-i) - 2c(c-i) - 3c+3i+1 \right) \\
&+ (2c-i)(2c-i-1)(x-c)^2 - (2c-i) \left(2c(c-i+1) - 6c+3i+1 \right) (x-c) \\
&+ (c-i) \left((c-i-1)c^2 - (3c-3i-1)c - 2(2c-2i-1)(c-i) \right) \\
&= i(i+1)x^2 - i(2(c-i)d + 2ci + 3i+1)x \\
&+ (c-i)(c-i-1)d^2 + (2ci + c - 3i - 1)(c-i)d + i(c^2i + c^2 - 3ci + 4i^2 - c + 2i) \\
&= h(x, d, c, i). \tag{5.27}
\end{aligned}$$

This proves $h(v-k, d, c, i) \geq 0$.

Next, let \mathcal{B}' be the block set of \mathcal{D}' . Then $\mathcal{B}' = \{\Omega \setminus B \mid B \in \mathcal{B}\}$. Let

$$S'_1 = \{X' \subset \Omega \mid |X'| = d' + 2(i' - 2), |X' \cap B'| \leq d' + i' - 2 \text{ for } \forall B' \in \mathcal{B}'\}.$$

Then we can easily show that the following conditions are equivalent.

- (1) $X' \in S'_1$.
- (2) $|X'| = d' + 2(i' - 2)$ and $|X' \cap B'| \leq d' + i' - 2$ for any block $B' \in \mathcal{B}'$.
- (3) $|(\Omega \setminus X) \cap (\Omega \setminus B)| \leq v - k - i - 1$ for $X = \Omega \setminus X'$ and any block $B \in \mathcal{B}$.
- (4) $|X| = d + 2(i + 1)$, $X = \Omega \setminus X'$ and $|X \cap B| \leq d + i + 1$ for any $B \in \mathcal{B}$.
- (5) $X \in S_2$.

Also the following conditions are equivalent.

- (6) $|X' \cap B'| = d' + i' - 2$ for $X' \in S'_1$ and a block $B' \in \mathcal{B}'$.
- (7) $|X \cap B| = d + i + 1$ for $X \in S_2$ and a block $B \in \mathcal{B}$.

Therefore Proposition 6 implies $h(v-k, d, c, i) = 0$ if and only if $|\{B \in \mathcal{B} \mid |X \cap B| = d + i - 1\}|$ does not depend on the choice of $X \in S_2$. This completes the proof of Proposition 7. ■

Finally we present a proof of Lemma 5.1.

Proof of Lemma 5.1.

First we assume $m \geq s$. Let Ω_1 be a $(m+j)$ point set. We count the cardinality of the following set in two ways.

$$\{(X, Y) \mid X, Y \subset \Omega_1, |X| = s, |Y| = j, X \cap Y = \emptyset\}. \tag{5.28}$$

For each $Y \subset \Omega_1$ with $|Y| = j$ there is $\binom{m}{s}$ choices for X . Therefore the cardinality of the set given in (5.28) equals $\binom{m+j}{j} \binom{m}{s}$. Next, we take a j point subset W of Ω_1 and fix

it. For a s point set $X \subset \Omega_1$, we have $0 \leq |X \cap W| \leq j$. The number of s point subsets X with $|X \cap W| = l$ is $\binom{j}{l} \binom{m}{s-l}$ for $0 \leq l \leq j$ and each X with $|X \cap W| = l$ the number of the choices of Y is $\binom{m+j-s}{j}$. Hence the cardinality of the set in (5.28) equals

$$\left(\binom{m}{s} + \binom{m}{s-1} \binom{j}{1} + \binom{m}{s-2} \binom{j}{2} + \cdots + \binom{m}{s-j} \binom{j}{j} \right) \binom{m+j-s}{j}. \quad (5.29)$$

Since $\binom{m+j-s}{j} > 0$, we have

$$\begin{aligned} \binom{m}{s} + \binom{m}{s-1} \binom{j}{1} + \binom{m}{s-2} \binom{j}{2} + \cdots + \binom{m}{s-j} \binom{j}{j} &= \frac{\binom{m+j}{j} \binom{m}{s}}{\binom{m+j-s}{j}} \\ &= \frac{\binom{m+j}{j} \binom{m}{s-j}}{\binom{s}{j}}. \end{aligned} \quad (5.30)$$

Next we assume $s > m \geq s - j$. Since $\binom{m}{s} = \binom{m}{s-1} = \cdots = \binom{m}{m+1} = 0$, the left side of (5.1) equals

$$\begin{aligned} \binom{m}{m} \binom{j}{s-m} + \binom{m}{m-1} \binom{j}{s-m+1} + \cdots + \binom{m}{s-j} \binom{j}{j} \\ = \binom{m}{0} \binom{j}{m+j-s} + \binom{m}{1} \binom{j}{(m+j-s)-1} + \cdots + \binom{m}{m+j-s} \binom{j}{0}. \end{aligned} \quad (5.31)$$

By the assumption we have $0 \leq m+j-s \leq j$. Then we consider the following set consisting of pairs of subset in $(m+j)$ point set Ω_1 .

$$\{(X, Y) \mid X, Y \subset \Omega_1, |X| = m+j-s, |Y| = j, X \cap Y = \emptyset\}. \quad (5.32)$$

We count the cardinality of the set defined in (5.32) in two ways. Then by a similar consideration we have

$$\binom{m}{0} \binom{j}{m+j-s} + \binom{m}{1} \binom{j}{m+j-s-1} + \cdots + \binom{m}{m+j-s} \binom{j}{0} = \frac{\binom{m+j}{j} \binom{m}{m+j-s}}{\binom{s}{j}}.$$

By (5.31) this implies (5.1). Finally if $s > m+j$, then both side of (5.1) equals 0. This completes the proof of Lemma 5.1. \blacksquare

6 Some Open Problems

Finally we present some open problems in $\beta(i)$ designs. The most fundamental problem is the following.

Problem 1. *Does there exist a $\beta(i)$ design which has the parameter (v, k, d) with $d > 0$ and $1 < i < k - d$ other than Steiner system $S(5, 8, 24)$ and the complementary design of*

it ?

The Steiner system $S(5, 8, 24)$ is both a $\beta(1)$ and a $\beta(2)$ design and the complementary design of it is both a $\beta(3)$ and a $\beta(4)$ design. Hauck [8] proved that if \mathcal{D} is both a $\beta(1)$ and a $\beta(2)$ design then \mathcal{D} is the Steiner system $S(5, 8, 24)$ or a Steiner system $S(t, t+1, 2t+2)$ for some t . So our second problem is the following.

Problem 2. *Does there exist a $\beta(i)$ design which is also a $\beta(i+1)$ design other than the Steiner system $S(5, 8, 24)$, the complementary design of it and Steiner systems $S(t, t+1, 2t+2)$?*

It is proved in [9] that there exists no $\beta(i)$ design which is also a $\beta(i+1)$ design if $i \equiv 2 \pmod{4}$. Therefore there exists no $\beta(2)$ design which is also a $\beta(3)$ design. In a forthcoming paper [3] we prove that if \mathcal{D} is both a $\beta(3)$ and a $\beta(4)$ design with the parameter (v, k, d) , then \mathcal{D} is the complementary design of the Steiner system $S(5, 8, 24)$ or possibly $v = 2k$ and $k = d + 6$. Generally $\beta(i)$ designs with the parameter $(v = 2k, k = d + 2i, d)$ achieve the lower bound of (1.6), whence are also $\beta(i+1)$ designs. In the case $i = 1$ such designs do exist. Our third problem is the following.

Problem 3. *Does there exist a $\beta(i)$ design which has the parameter $(v = 2k, k = d + 2i, d)$ for $i > 1$?*

Whether there exists a perfect e -code \mathcal{C} in $J(v, k)$ with $|\mathcal{C}| \geq 2$ and $1 < d(\mathcal{C}) < k$ is a well known open problem. This is stated in terms of a $\beta(i)$ design as follows.

Problem 4. *Does there exist a $\beta(i)$ design which has the parameter $(v, k = d + 2i - 1, d)$ with $0 < d < k - 1$?*

Acknowledgments

The authors thank Eiichi Bannai for valuable suggestions especially telling us the relation between $\beta(i)$ designs and perfect codes in $J(v, k)$, and introducing the work of Etzion-Schwartz [7] and Ahlswede-Aydinian-Khachatrian [2]. We also would like to express our gratitude to the referee for helpful comments and advice.

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